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CAPILLARY SURFACE CONTINUITY ABOVE IRREGULAR DOMAINS

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MATHEMATICS RESEARCH CENTER

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ABSTRACT

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There are simple domains $\cancel{\Omega}$ in $\cancel{R^2}$ having re-entrant corners for which

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the variational solution u to the capillary problem exists, is bounded, but

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does not extend continuously to $\partial\Omega$. How possible is it to characterize

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domains for which u must extend continuously? This paper contains the following answer:

<u>Theorem</u> let $P_0 \in \partial\Omega$. Let u be the (variational) solution to the capillary problem in Ω with *contact angle* γ, $0 < \gamma < \pi/2$. If any of (i), (ii), or (iii) hold, then u extends continuously to P_0:

(i) Ω is convex in a neighborhood of P_0, $\partial\Omega$ has local Lipschitz constant $L_1 < \tan\gamma$.

(ii) $\partial\Omega$ is (locally) two C^1 curves meeting at P_0 with interior angle θ satisfying $\pi - 2\gamma < \theta < \pi$.

(iii) $\partial\Omega$ is C^1 at P_0.

AMS(MOS) Subject Classification: 35J20, 35J25.
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SIGNIFICANCE AND EXPLANATION

After water is poured into a vertical tube with horizontal cross section Ω the function u whose graph S_u describes the meniscus height satisfies a nonlinear elliptic partial differential equation: The mean curvature of S_u is proportional to its height above a fixed reference plane, and at the boundary the angle of contact between (the lower normal of) S_u and (the exterior normal of) $\partial\Omega \times \mathbb{R}$ is physically determined (constant if the cylinder is of uniform contruction).

Nonlinear elliptic equations with co-normal derivative boundary conditions, of which the above capillary problem with prescribed contact angle is a prime example, occur frequently in physical problems but have not been as extensively studied as those with Dirichlet boundary conditions (u prescribed on $\partial\Omega$). There are many known results about when variational (weak) solutions to Dirichlet problems extend continuously to the boundary, but there is not much literature about the same question for co-normal derivative problems. That is the problem studied in this paper--for the special case of a two dimensional capillary surface making prescribed angle of contact with the bounding cylinder.

Roughly speaking, the main theorem in this paper says that u extends continuously to a point $P_0 \in \partial\Omega$ if $\partial\Omega$ is C^1 there, if $\partial\Omega$ is the vertex of two C^1 curves meeting with a convex (less than π) interior angle there, or if $\partial\Omega$ is convex with a not too large Lipschitz constant there. Since there are easy counterexamples to show that u need not extend continuously if P_0 is the vertex of a re-entrant corner (interior angle greater than π), these conditions are almost necessary as well as being sufficient. Because well known results show that u actually extends smoothly wherever $\partial\Omega$ is sufficiently smooth, the importance of these results is that they require a minimal amount of regularity for $\partial\Omega$.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

CAPILLARY SURFACE CONTINUITY ABOVE IRREGULAR DOMAINS

Nicholas J. Korevaar

§1. Introduction

For a Lipschitz domain Ω in $\mathbb{R}^2$ a function $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a classical solution to the capillary problem in a gravitational field if:

(1) $\operatorname{div} Tu = 2H(S_u) = \kappa u$ in Ω,

$Tu = \dfrac{Du}{\sqrt{1 + |Du|^2}}$, $H(S_u)$ = mean curvature of the graph S_u of u, $\kappa > 0$.

(2) $Tu \cdot n = \cos \gamma$ on $\partial\Omega$,

$0 < \gamma < \pi$ prescribed, n = exterior normal to $\partial\Omega$.

Physically, S_u describes the capillary surface formed when a vertical cylinder with horizontal cross section Ω is placed in an infinite reservoir of liquid having rest height equal to zero.

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Then

$\kappa = \frac{\rho g}{\sigma}$ where ρ = density of liquid

g = (downward) acceleration of gravity

σ = surface tension between liquid and air

$\cos\gamma = \frac{\sigma_1}{\sigma}$ where σ_1 = surface attraction between liquid and cylinder.

Geometrically, γ is the contact angle between the (downward normal to the) capillary surface S_u and the (exterior normal to the) bounding cylinder $\partial\Omega \times \mathbb{R}$.

The function u minimizes the variational problem associated with (1,2) if it minimizes the energy functional

$$E(f) = \int_\Omega \left(\sqrt{1 + |Df|^2} + \frac{\kappa}{2} f^2\right) - \int_{\partial\Omega} \nu f, \quad \nu = \frac{\sigma_1}{\sigma} = \cos\gamma \tag{3}$$

over the appropriate space of functions.

There are simple domains Ω in $\mathbb{R}^2$ having re-entrant corners for which the variational solution to the capillary problem exists, is bounded, but does not extend continuously to $\partial\Omega$ [13]. How possible is it to characterize domains for which u must extend continuously?

Theorem. Let $P_0 \in \partial\Omega$. Let u be the (variational) solution to the capillary problem in Ω with contact angle γ, $0 < \gamma \leq \pi/2$. If any of (i), (ii), or (iii) hold, then u extends continuously to P_0:

(i) Ω is convex in a neighborhood of P_0, $\partial\Omega$ has local Lipshitz constant $L_1 < \tan\gamma$;

(ii) $\partial\Omega$ is (locally) two C^1 curves meeting at P_0 with interior angle θ satisfying $\pi - 2\gamma < \theta < \pi$;

(iii) $\partial\Omega$ is C^1 at P_0.

Since well known boundary regularity results imply that u extends smoothly wherever $\partial\Omega$ is smooth enough (see e.g. [17, 19]), the importance of this theorem is that it does not require much regularity for $\partial\Omega$. Roughly speaking, in order for u to have a jump at P_0 it needs room, such as at a re-entrant corner. If the corner is convex or there is no corner, a bounded u can have no jump and must be continuous.

Studying the same problem, Simon has shown that if P_0 is the vertex of two $C^{1,\alpha}$ curves meeting with interior angle θ, $\pi - 2\gamma < \theta < \pi$, then u actually extends to be C^1 at P_0 [16]. His method requires a strict corner $\theta < \pi$, $C^{1,\alpha}$ curves, and uses geometric measure theory to get a strong result: u is C^1 at P_0. Our method requires less of the boundary, uses only comparison methods, but gives a weaker result: u is continuous at P_0.

(The condition $\theta > \pi - 2\gamma$ (and the related $L_1 < \tan\gamma$) that occurs in these theorems is essentially geometric: If u was to extend to be C^1 at P_0 and if the contact angle condition was to be satisfied with both faces of $\partial\Omega \times R$ there, then $\theta > \pi - 2\gamma$ is necessary for the tangent plane to exist. For $\theta < \pi - 2\gamma$ *no* tangent plane can exist, and in fact u approaches infinity as P_0 is approached in this case [3]).

§2. Preliminaries

First some notation: Domains Ω shall be bounded with Lipschitz boundary $\partial\Omega$ unless otherwise stated. We restrict to the case $\Omega \subset \mathbb{R}^2$. All integrals over a domain are with respect to (2-dimensional) Lebesgue measure, all integrals over boundaries of domains are with respect to (1-dimensional) Hausdorff measure. $|\Omega|$ is the area of Ω, $|\partial\Omega|$ is the length of $\partial\Omega$. $B_\rho(P_0)$ and $S_\rho(P_0)$ are the open ball of radius ρ centered at P_0 and its boundary sphere.

We discuss capillary surfaces that solve the variational problem (3) associated with (1) and (2); u should minimize

$$\tilde{E}(f) = \int_\Omega \left(\sigma\sqrt{1 + |Df|^2} + \frac{\rho g}{2} f^2\right) - \int_{\partial\Omega} \sigma_1 f$$

over the appropriate space of functions. The three terms making up the energy functional are (in order) surface energy, potential energy from gravity, and wetting energy. Emmer [5] and Finn-Gerhardt [7] have studied the existence of variational solutions u to the capillary problem in Lipshitz domains Ω. When it exists the function u is real analytic in Ω and satisfies (1). Wherever $\partial\Omega$ is smooth enough u extends smoothly and satisfies the boundary condition (2) classically [17, 19].

Even when $\partial\Omega$ is only Lipschitz, however, the fact that u minimizes the energy (3) implies that (2) is satisfied in some weak (integral) sense. This weak boundary behavior is still strong enough to use a comparison method for surfaces of related

mean curvature and contact angle. The comparison method has been widely used in capillarity [1-4, 7, 13-15, 20, 22-24] and a version of it is also the primary tool used to prove our main theorem.

We state below the particular form of the comparison principle used here, after defining the type of weak boundary behavior for which it is valid.

Definition. A sequence of domains $\{\Omega_k\}$ exhausts Ω if each $\partial\Omega_k \in C^1$, $\bar{\Omega}_k \subset \Omega_{k+1}$ and $\cup_{k=1}^{\infty} \Omega_k = \Omega$.

Definition. Let $u \in C^2(\Omega)$, $\nu \in L^{\infty}(\partial\Omega)$. $Tu \cdot n = \nu$ weakly on $\partial\Omega$ if for any exhausting sequence $\{\Omega_k\}$, and for any $f \in L^{\infty}(\Omega) \cap W^{1,1}(\Omega)$,

$$\lim_{k\to\infty} \int_{\partial\Omega_k} f Tu \cdot n_k = \int_{\partial\Omega} \nu f \quad n_k = \text{exterior normal to } \partial\Omega_k.$$

Local boundary values of $Tu \cdot n$ must sometimes be considered. This can be done in a manner consistent with the above definition:

Definition. Let $U \subset R^n$ be open. Let $u \in C^2(\Omega)$, $\nu \in L^{\infty}(\partial\Omega \cap U)$. $Tu \cdot n = \nu$ weakly on $\partial\Omega \cap U$ if for any exhausting sequence $\{\Omega_k\}$ of Ω and any $f \in L^{\infty}(\Omega) \cap W^{1,1}(\Omega)$ s.t. the essential support of the trace of f on $\partial\Omega$ is compactly contained in $U \cap \partial\Omega$,

$$\lim_{k\to\infty} \int_{\partial\Omega_k} f Tu \cdot n_k = \int_{\partial\Omega} \nu f.$$

Remark. To prove that the variational solution satisfies $Tu \cdot n = \nu$ weakly, one does the usual Euler-Lagrange derivation of equations and boundary conditions from (3), except that one integrates by parts over Ω_k instead of over Ω (u is smooth in Ω, hence over $\bar{\Omega}_k$). The details can be found in [7], or for a more systematic exposition, [14]. In the same references it is also shown that a smooth surface S_v that classically satisfies $Tv \cdot n = \nu$ (wherever n is defined) also satisfies $Tv \cdot n = \nu$ weakly.

Comparison Principle. (Essentially Lemma 3 of [7]) Let $v, w, \in C^2(\Omega)$. Consider a component $\mathcal{O}$ of the set on which $v > w$. Suppose that:

(i) $x \in \mathcal{O} \Rightarrow \operatorname{div} Tv(x) > \operatorname{div} Tw(x)$,

(ii) on $\bar{\mathcal{O}} \cap \partial\Omega$, $Tv \cdot n \leq Tw \cdot n$ weakly.

Then $\mathcal{O} = \phi$. (So if (i) and (ii) are true for all components, $v \leq w$).

(Condition (ii) means that there is some open set $\mathcal{U} \supset \bar{\mathcal{O}} \cap \partial\Omega$ on which $Tv \cdot n$ and $Tw \cdot n$ are defined weakly and that their weak values satisfy $Tv \cdot n \leq Tw \cdot n$ almost everywhere on $\bar{\mathcal{O}} \cap \partial\Omega$). See [7] or [14] for the proof of the comparison principle.

§3. Proof of Theorem

The techniques used to prove the main theorem are barehanded. The accompanying figures should help keep notation and ideas organized. The proof follows this outline:

We consider the domain Ω, $P_0 \in \partial\Omega$, and the variational solution $u \in C^2(\Omega) \cap H^{1,1}(\Omega)$ of (3). We take γ to be constant, $0 < \gamma < \pi$. (Essentially the same proof works if γ is allowed to be a continuous function on $\partial\Omega$). Since u minimizes (3) iff $-u$ minimizes (3) with ν replaced by $-\nu$, it is no loss of generality to assume $0 < \gamma \leq \pi/2$, and we do.

Let

$$U = \limsup_{P \to P_0,\ P \in \Omega} u(P), \quad L = \liminf_{P \to P_0,\ P \in \Omega} u(P).$$

If L and U are finite, then certain geometric contraints on $\partial\Omega$ imply the existence of "ridges" and ditches": For any $\varepsilon > 0$, $\exists$ $\rho_1 > 0$ so that for any radius ρ, $0 < \rho < \rho_1$, there is a ridge $\mathcal{R}_\rho^\varepsilon$ and a ditch $\mathcal{D}_\rho^\varepsilon$. These are connected components in $B_\rho(P_0) \cap \Omega$ of the sets on which $u(P) > U - \varepsilon$ and $u(P) < L + \varepsilon$, respectively, and extend from P_0 to $S_\rho(P_0)$. (See Figure 2 and Lemma 3).

For domains satisfying conditions (i), (ii), or (iii) of the main theorem an explicit comparison function is constructed (Figure 3 and Lemma 4). If $L \neq U$, it is used to slice through either the ridge or the ditch and thus to contradict the comparison principle (Figures 3, 4, and Theorem 1). Thus, $L = U$ and u is continuous at P_0.

The existence of ditches and ridges as well as the argument used with the comparison principle depend on the special topology of $\mathbb{R}^2$. The comparison argument is similar in spirit to one used by Finn for minimal surfaces [6], and later by Finn-Giusti for constant mean curvature surfaces [8] to obtain interior gradient bounds in two dimensions.

Because $\partial\Omega$ is Lipschitz near P_0 there is a Lipschitz function $y = \phi(x)$ and a neighborhood V of the origin so that after a translation and rotation

$$P_0 = (0,0)$$

$$V \cap \Omega = V \cap \{p = (x,y) \ \text{ s.t. } \ y<\phi(x)\}. \tag{4}$$

If Ω satisfies (i), (ii), or (iii) of the theorem stated in the introduction, we may also suppose that

$$\lim_{x\to 0^-} \phi'(x) > 0 > \lim_{x\to 0^+} \phi'(x) \tag{5}$$

and

$$\sup_{(x,\phi'(x)) \in V} |\phi'(x)| = L_1 < \tan\gamma \tag{6}$$

(The limits and sup are taken over those values of x for which $\phi'(x)$ exists).

We prove:

Theorem 1. If (4)-(6) are satisfied and if u is the variational solution to the capillary problem with contact angle γ in Ω, then u extends continuously to P_0.

The first step is:

__Lemma 1__. If (4)-(6) are satisfied, there is a neighborhood $\mathcal{U} \subset \mathcal{V}$ of P_0 so that u is uniformly bounded in $\mathcal{U} \cap \Omega$.

__Proof__. We introduce the concept of an internal sphere condition with contact angle (due to Finn-Gerhardt [7]). Domains satisfying this condition have uniformly bounded solutions u. We then show that if Ω satisfies (4)-(6) of Section 3, it satisfies the appropriate internal sphere condition with contact angle γ, near P_0.

__Definition__. Let $0 < \gamma_0 < \pi/2$ and $P \in \Omega$. P satisfies an internal sphere condition with contact angle γ_0 and radius ρ ($P \in \text{i.s.c.}_{\rho,\gamma_0}$) if there is a ball B_ρ of radius ρ, $P \in \bar{B}_\rho$, such that any lower hemisphere lying above $\bar{B}_\rho$ makes contact angle $< \gamma_0$ at all points of contact with $\partial\Omega \times \mathbb{R}$ for which the normal to $\partial\Omega$ is defined.

__Definition__. A neighborhood $\mathcal{U} \cap \Omega$ satisfies a uniform internal sphere condition with contact angle γ_0 and radius δ ($\mathcal{U} \cap \Omega \subset \text{I.S.C.}_{\delta,\gamma_0}$) if $P \in \mathcal{U} \cap \Omega \Rightarrow \exists\ \rho > \delta$ s.t. $P \in \text{i.s.c.}_{\rho,\gamma_0}$.

This concept is illustrated in Figure 4. We have:

__Step 1([7])__. If $\mathcal{U} \cap \Omega \subset \text{I.S.C.}_{\delta,\gamma_0}$, then any solution u to the variational capillary problem (3) with $|\nu| < \cos\gamma_0$ is uniformly bounded in $\mathcal{U} \cap \Omega$.

__Proof__. (See [7]). This is a straightforward application of the comparison principle. The comparison surfaces are the graph of the solution, sufficiently high lower hemispheres (for the upper bound) and sufficiently low upper hemispheres (for the lower bound).

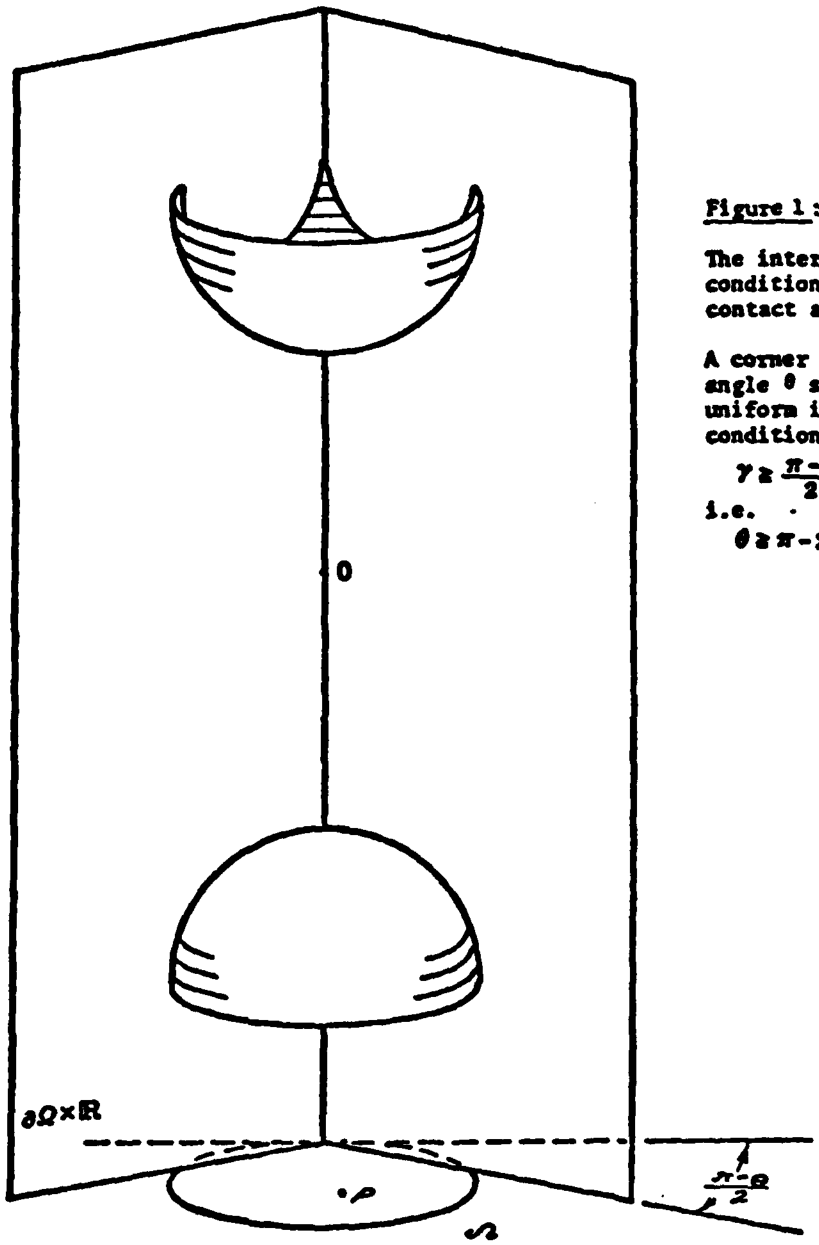

Figure 1:

The internal sphere condition with contact angle.

A corner with interior angle θ satisfies a uniform internal sphere condition for any

$$\gamma \geq \frac{\pi-\theta}{2},$$

i.e.

$$\theta \geq \pi-2\gamma.$$

Step 2. We show that if Ω satisfies (4)-(6) there is a neighborhood $\mathcal{U}$ of P_0 and a $\delta > 0$ so that $\mathcal{U} \cap \Omega$ I.S.C.$_{\delta,\gamma}$. Let $\varepsilon, \rho > 0$. Take

$$\mathcal{U} = B_\rho(P_1), \quad P_1 = (0, -\rho(1-\varepsilon)).$$

For small enough ε, ρ, we will have $\mathcal{U} \cap \Omega \subset$ I.S.C.$_{\delta,\gamma}$:

For the lower hemisphere above $\mathcal{U}$, $z = v(P)$, we have

$$Tv(P) = \frac{1}{\rho}(P - P_1).$$

The exterior normal n to $\partial\Omega$ is

$$n = \frac{1}{\sqrt{1 + (\phi')^2}}(-\phi', 1)$$

so that for $P = (x,y) \in B_\rho(P_1) \quad \partial\Omega$,

$$Tv \cdot n = \frac{1}{\rho\sqrt{1 + (\phi')^2}}(-x\phi' + y + \rho(1-\varepsilon)) \tag{7}$$

If $\phi'(x)$ exists then

$$y = \int_0^x \phi'(t)dt = \int_0^x (\phi'(t) - \phi'(x))dt + x\phi'(x).$$

But (5) implies that the right and left-hand limits of ϕ' exist at $x = 0$. Thus, for any $\varepsilon_1 > 0 \; \exists \; \rho_0 > 0$ s.t.

$$\rho < \rho_0 \Rightarrow y - x\phi'(x) > -\varepsilon_1\rho. \tag{8}$$

Combining (7) and (8) implies that if $\rho < \rho_0$,

$$Tv \cdot n > \frac{1 - \varepsilon - \varepsilon_1}{\sqrt{1 + (\phi')^2}}. \tag{9}$$

From (6) we have

(10) $$\frac{1}{\sqrt{1+(\phi')^2}} > \frac{1}{\sqrt{1+L_1^2}} > \frac{1}{\sqrt{1+\tan^2\gamma}} = \cos\gamma,$$

so that we may choose ε, ε_1 small enough to make

$$\frac{1}{\sqrt{1+(\phi')^2}}(1-\varepsilon-\varepsilon_1) > \cos\gamma.$$

Thus. for $\rho < \rho_0(\varepsilon_1)$,

$$Tv \cdot n > \cos\gamma. \qquad \text{Q.E.D.}$$

We use the following facts often and group them as:

Lemma 2. Let U be an open set, $v \in C^2(U) \cap H^{1,1}(U)$. Then for almost all z, $\{P: v(P) = z\}$ is a regular set (a collection of simple closed curves and curves without endpoints in U) with finite total length.

If Γ is any smooth curve in U, then almost all level sets of v in U are transversal to Γ.

Proof. Almost all level sets are regular by Sard's Theorem. The co-area fomula [9] implies that almost all the lengths are finite, since $v \in W^{1,1}(U)$.

Parameterize Γ by arclength s and consider the restriction of v to Γ. By Sard's theorem almost all level sets S are regular with $v'(s) \neq 0$ on S. These sets S must arise from transverse intersections between Γ and level sets of v in U. Q.E.D.

Lemma 3. If u is bounded and (6) is satisfied (in particular, if (4)-(6) are satisfied), then given $\varepsilon > 0$, $\exists\ \rho_1 > \rho$ s.t. for all $0 < \rho < \rho_1$,

(a) There is a ridge R_ρ^ε, a connected component of the set on which $u > U - \varepsilon$ satisfying $P_0 \in \bar{R}_\rho^\varepsilon$ and $\bar{R}_\rho^\varepsilon \cap S_\rho(P_0) \cap \Omega \neq \phi$.

(b) There is a ditch $\mathcal{D}_\rho^\varepsilon$, a connected component of the set on which $u < L + \varepsilon$ satisfying $P_0 \in \bar{\mathcal{D}}_\rho^\varepsilon$ and $\bar{\mathcal{D}}_\rho^\varepsilon \cap S_\rho(P_0) \cap \Omega \neq \phi$.

Proof. Since u is bounded in $\mathcal{U} \cap \Omega$, $H(S_u)$ is bounded (1) and we may pick $M_1 < \infty$ so that

(11) $$|\operatorname{div} Tu(P)| < M_1 \qquad P \in \mathcal{U} \cap \Omega.$$

Let $\varepsilon > 0$ be given. To find R_ρ^ε we compare S_u to the upper half of a horizontally inclined cylinder. Its axis of symmetry is parallel to the x-axis and it has radius $\frac{1}{M_1}$ (see Figure 2). Specifically, let

$$v(x,y) = \sqrt{\left(\frac{1}{M_1}\right)^2 - \left(y - \frac{1}{M_1} + \delta\right)^2} \qquad \delta > 0.$$

By a useful abuse of notation we continue to call this function v and its graph S_v, even if we raise it or lower it. By construction, for all P in the domain of v and in $\mathcal{U} \cap \Omega$,

(12) $$\operatorname{div} Tv = 2H(S_v) = -M_1 < \operatorname{div} Tu.$$

Making δ small enough and picking a small enough neighborhood $\mathcal{U}_1 \subset \mathcal{U}$ of P_0 we can satisfy

(13) $$Tv \cdot n > \cos \gamma \quad \text{on} \quad \partial\Omega \cap \mathcal{U}_1 .$$

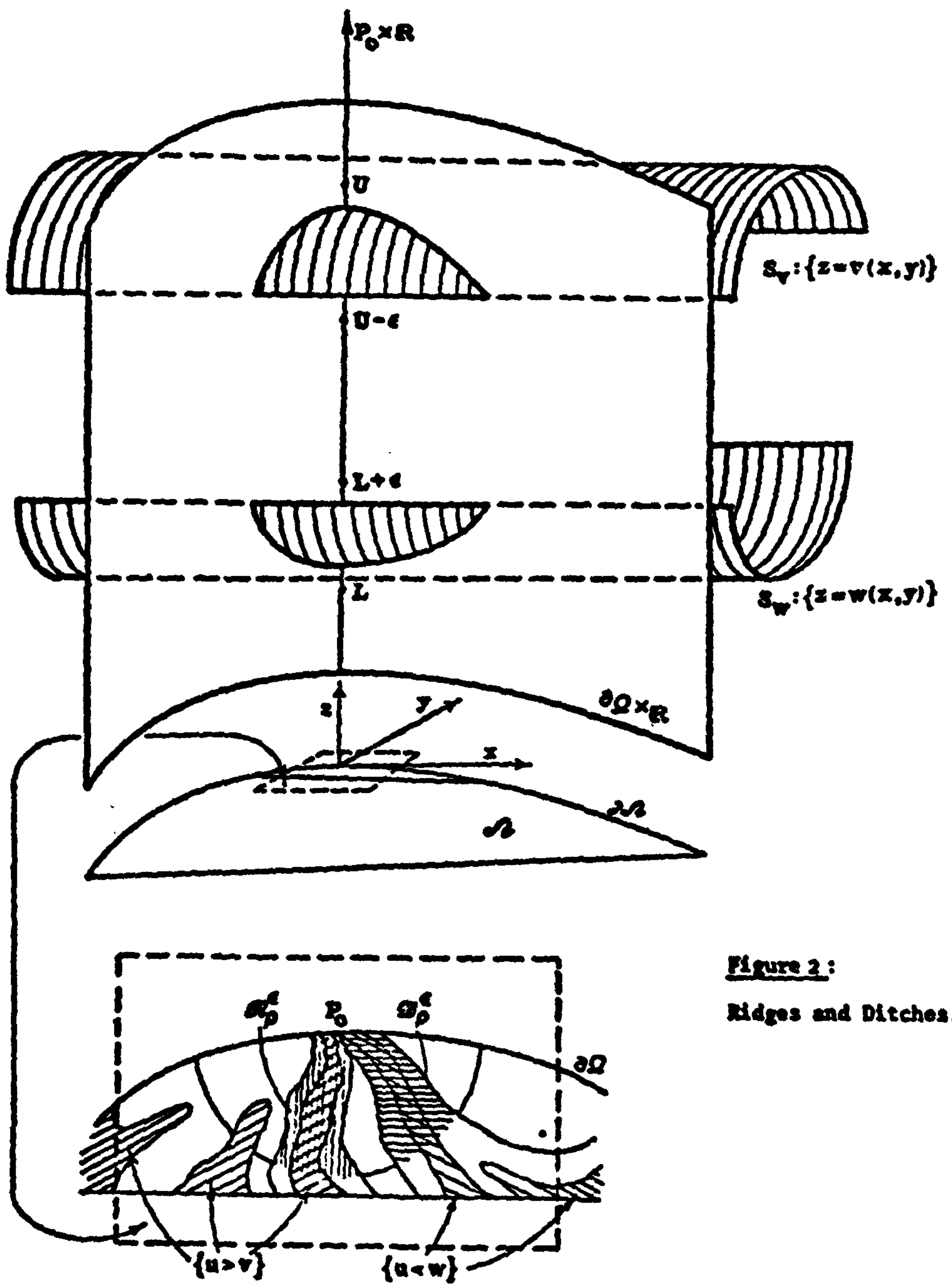

Figure 2:

Ridges and Ditches.

This is because $L_1 < \tan \gamma$:

$$Tv \cdot n = \frac{v_y}{\sqrt{1 + v_y^2}} \frac{1}{\sqrt{1 + \phi'^2}} > \frac{v_y}{\sqrt{1 + v_y^2}} \frac{1}{\sqrt{1 + L_1^2}} > \frac{v_y}{\sqrt{1 + v_y^2}} \cos \gamma.$$

(By making δ small enough, $\frac{v_y}{\sqrt{1 + v_y^2}}$ can be made arbitrarily close to 1 near $P_0 = (0,0)$).

Lift S_v so that the points P in $\mathcal{U}_1 \cap \Omega$ above which S_v intersects S_u from a regular set with finite total length (Lemma 2), and so that

$$U - \frac{2\epsilon}{3} < v(P_0) < U - \frac{\epsilon}{3}.$$

Since v is continuous at P_0 there is a $\rho_1 > 0$, $B_{\rho_1}(P_0) \subset \mathcal{U}_1$ and

(14) $$U - \frac{3}{4}\epsilon < v(P) < U - \frac{\epsilon}{4} \quad P \in B_\rho(P_0)$$

For $\rho < \rho_1$, (12), (13), and the comparison principle imply that every component $\mathcal{O}$ in $B_\rho \cap \Omega$ of the set on which $u > v$ must have limit points intersecting S_ρ (in a set of non-zero Hausdorff measure).

Because of (14) there are points P arbitrarily near P_0 where $u(P) > v(P)$. For each $i \in \mathbb{N}$ (with $1/i < \rho$) pick P_i s.t.

$$|P_0 - P_i| < \frac{1}{i}, \quad u(P_i) > v(P_i).$$

Let $\mathcal{O}_i$ be the component in $B_\rho \cap \Omega$ of the set on which $u > v$ containing P_i. Since $\bar{\mathcal{O}}_i \cap S_\rho(P_0) \neq \phi$, it follows that

$$(15) \qquad |\partial \mathcal{O}_i| > 2|\rho - \frac{1}{i}|.$$

Since $\partial(\bigcup_i \mathcal{O}_i)$ has finite length in B_ρ and since at most two $\mathcal{O}_i$'s can share any given boundary curve, there are only a finite number of distinct $\mathcal{O}_i$'s. Hence, one of them, $\mathcal{O}_{i_1}$, contains infinitely many P_i's, so has P_0 as a limit point. From (14) it follows that $\mathcal{O}_{i_1}$ is part of a larger component in $B_\rho \cap \Omega$ of the set on which $u > U - \varepsilon$. This component $\mathcal{R}_\rho^\varepsilon$ is our ridge.

Finding the ditch $\mathcal{D}_\rho^\varepsilon$ uses the analogous procedure: Instead of the upper half of the cylinder placed to intersect $P_0 \times R$ at a height just beneath U, one places the lower half of the cylinder so that it intersects $P_0 \times R$ just above L (see Figure 1). Q.E.D.

<u>Remark</u>. Lemma 3 uses the fact that $\Omega \subset R^2$ crucially; the inequality (15) would not follow in general R^n.

<u>Lemma 4</u>. Suppose (4)-(6) are satisfied. Let the following be given

$$\delta_1, \delta_2 > 0$$

$$L, U, M \text{ satisfying } -M < L < U < M.$$

Then there is a neighborhood $\mathcal{U}$ of $P_0 = (0,0)$ and a comparison function v defined on $\mathcal{U} \cap \Omega$ satisfying

(i) $v \in C^2(\mathcal{U} \cap \Omega)$, $|\mathrm{div}\, Tv - \kappa v| < \kappa\delta_1$;

(ii) $\partial(\mathcal{U} \cap \Omega) = \bigcup_{i=1}^{4} \Gamma_i$, where the Γ_i are described in

Figure 3, and

$$\begin{aligned} Tv\cdot n &> \cos\gamma && \text{on } \Gamma_1 \\ v &= M && \text{on } \Gamma_2 \\ v &= -M && \text{on } \Gamma_3 \\ Tv\cdot n &< \cos\gamma && \text{on } \Gamma_4; \end{aligned}$$

(iii) For P near enough P_0, $|v(P) - \frac{L+U}{2}| < \delta_2$.

Proof. (see Figure 3) S_v will be almost vertical and its level sets at height z will almost be arcs with curvature κz. (The fact that almost vertical surfaces with almost circular level sets can almost satisfy the capillary equation has been used effectively to study capillary surface behavior above corners with interior angle $\theta < \pi - 2\gamma$ [3]).

The function $z = v(x,y)$ is given implicitly (after a rotation about the z-axis) by

(16) $$x = \frac{\kappa z}{2} y(y+h) + \delta(y+h)\varphi(z)$$

where h is small, $\delta \ll h$, and $\varphi(z)$ satisfies

$$|\varphi|_{C^2[-M,M]} < 1,\ \frac{\partial\varphi}{\partial z} < 0,\ \varphi = 0 \text{ in a small neighborhood of } z = \frac{L+U}{2}.$$

If $\delta = 0$, the surface S_v has a level set at height z that is part of the parabola through $(0,0)$ and $(0,-h)$, and that has curvature almost equal to κz. For $0 < \delta \ll h$ the second term on the right of (16) is a small perturbation added so that v satisfies (ii) and (iii).

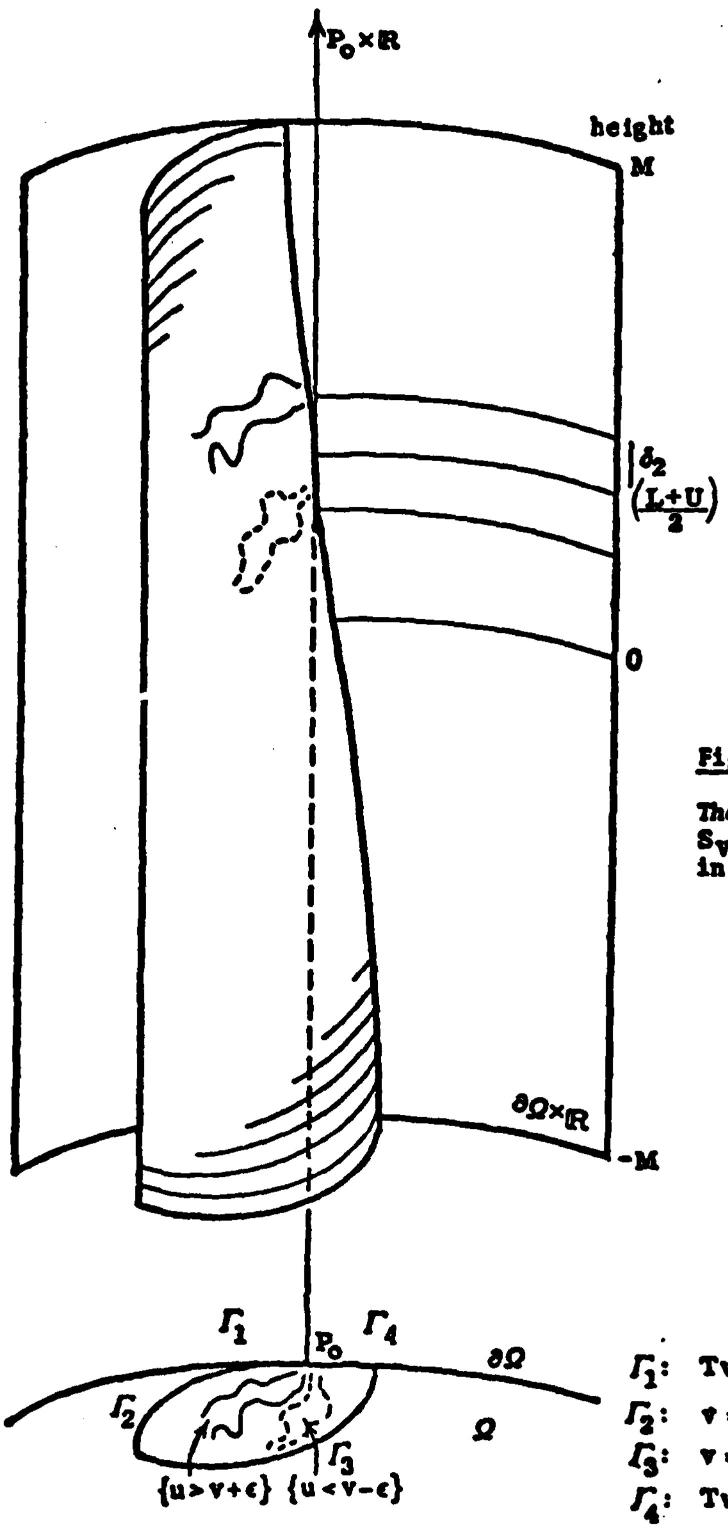

Figure 3:

The comparison surface S_v and its projection in $\mathbb{R}^2$.

Figure 2 shows the surface S_v satisfying all three conditions of Lemma 2. The key property of the differential equation (1) that allows the construction of S_v is that the mean curvature is strictly increasing with height. The key property of $\partial\Omega$ is the "half C^1" condition (5). (It's necessary for (ii)). When it doesn't hold, for example at a re-entrant corner, u may fail to be continuous [13].

The calculations verifying (i), (ii), and (iii) of Lemma 4 are included as Appendix A. Q.E.D.

Lemma 5. (A crossing lemma in two dimensional topology)

Consider a simple closed curve Σ with interior $\mathcal{O}$. Let A, B, C, D be a distinct, consecutively-ordered points on Σ. If a simple curve Γ in $\mathcal{O}$ has A and C as its two endpoints, then the following hold:

(i) There is no curve in $\mathcal{O}$, disjoint from Γ, connecting B to D.

(ii) No connected open subset of $\mathcal{O}$ disjoint from Γ can have both B and D as limit points.

Proof. This lemma is a consequence of the Jordan Curve Theorem: Consider the simple curve obtained by following Γ from A to C, then following Σ from C to D to A. All points in a neighborhood $\mathcal{B}$ of B are exterior to this curve. (They can be connected to infinity). All points in a neighborhood $\mathcal{D}$ of D and inside $\mathcal{O}$ are interior to the curve (since the points outside $\mathcal{O}$ are exterior to it). Hence, any

path in $\mathcal{O}$ connecting points in $\mathcal{D} \cap \mathcal{O}$ to points in $\mathcal{B} \cap \mathcal{O}$ must cross Γ. This proves (i) and (ii). Q.E.D.

We are now in a position to prove Theorem 1. We suppose $U > L$, equations (4), (5), (6), and get a contradiction. Let

$$\varepsilon < \frac{U-L}{6}. \tag{17}$$

By Lemma 3, $\exists\ \rho_1 > 0$ s.t. for $\rho < \rho_1$ there is a ridge $\mathcal{R}_\rho^\varepsilon$ and a ditch $\mathcal{D}_\rho^\varepsilon$. Pick

$$R_\rho \in \bar{\mathcal{R}}_\rho^\varepsilon \cap S_\rho(P_0) \cap \Omega \qquad D_\rho \in \bar{\mathcal{D}}_\rho^\varepsilon \cap S_\rho(P_0) \cap \Omega \tag{18}$$

We assume that there are arbitrarily small $\rho > 0$ for which R_ρ is oriented clockwise in Ω from D_ρ, on $S_\rho(P_0)$. This is no loss of generality: One could reflect the domain Ω and the solution u across the y-axis, preserving (4), (5), (6), and the resulting lemmas, but changing the relative orientations of D_ρ and R_ρ. Let

$$\delta_1 = \delta_2 = \varepsilon. \tag{19}$$

Pick a comparison surface S_v satisfying Lemma 4. $\mathcal{U}$ will be the domain from this lemma, above which v is defined. From (17)-(19), and (iii) of Lemma 4, $\exists\ \rho_2 < \rho_1$ s.t. for $\rho < \rho_2$,

$$\begin{aligned} P \in \bar{\mathcal{R}}_\rho^\varepsilon \cap \Omega &\Rightarrow u(P) > U - \varepsilon > \frac{U+L}{2} + 2\varepsilon > v(P) + \varepsilon \\ P \in \bar{\mathcal{D}}_\rho^\varepsilon \cap \Omega &\Rightarrow u(P) < L + \varepsilon < \frac{U+L}{2} - 2\varepsilon < v(P) - \varepsilon. \end{aligned} \tag{20}$$

Fix a $\rho < \rho_2$ and the resulting R^{ε}_{ρ}, $\mathcal{D}^{\varepsilon}_{\rho}$, R_{ρ}, D_{ρ}, with R_{ρ} oriented clockwise in Ω from D_{ρ}, on $S_{\rho}(P_0)$. Equations (18), (20) imply that $\bar{R}^{\varepsilon}_{\rho} \cap \Omega$ is contained in a component $\mathcal{R}$ in $\mathcal{U} \cap \Omega$ of the set on which $u > v + \varepsilon$. Also, $\bar{\mathcal{D}}^{\varepsilon}_{\rho} \cap \Omega$ is contained in $\mathcal{D}$, a component in $\mathcal{U} \cap \Omega$ of the set on which $u < v - \varepsilon$. Combining (i) of Lemma 4 with (19), (20) yields

$$
(21) \quad \begin{aligned}
&\text{for points in } \mathcal{R},\ \operatorname{div} T(v+\varepsilon) = \operatorname{div} Tv < \kappa(v+\delta_1) < \kappa u = \operatorname{div} Tu \\
&\text{for points in } \mathcal{D},\ \operatorname{div} T(v-\varepsilon) > \operatorname{div} Tu.
\end{aligned}
$$

Thus, the combination of the comparison principle and all four conditions from (ii) of Lemma 4 imply

$$
(22) \quad \begin{aligned}
&|\bar{\mathcal{R}} \cap (\Gamma_3 \cup \Gamma_4)| \neq 0 \\
&|\bar{\mathcal{D}} \cap (\Gamma_1 \cup \Gamma_2)| \neq 0.
\end{aligned}
$$

Because of the way they were picked, (22) implies that $\mathcal{R}$ and $\mathcal{D}$ must "cross" each other (see Figure 4). But $\mathcal{R}$ and $\mathcal{D}$ cannot intersect. Making this argument rigorous is technically tedious, but proves Theorem 1.

We can assume that $\partial\mathcal{R}$, $\partial\mathcal{D}$ are regular, of finite length in $\mathcal{U} \cap \Omega$, and that $\partial\mathcal{R}$ and $\partial\mathcal{D}$ meet $S_{\rho}(P_0)$ transversally. (If not, use Lemma 1 and pick a slightly larger $\varepsilon < \frac{L+U}{6}$. The old $\mathcal{R}, \mathcal{D}$ will be contained in a new $\mathcal{R}$ and $\mathcal{D}$, (larger) components of the sets on which $u > v + \varepsilon$, $U < v - \varepsilon$, respectively).

We show that (22) cannot be satisfied.

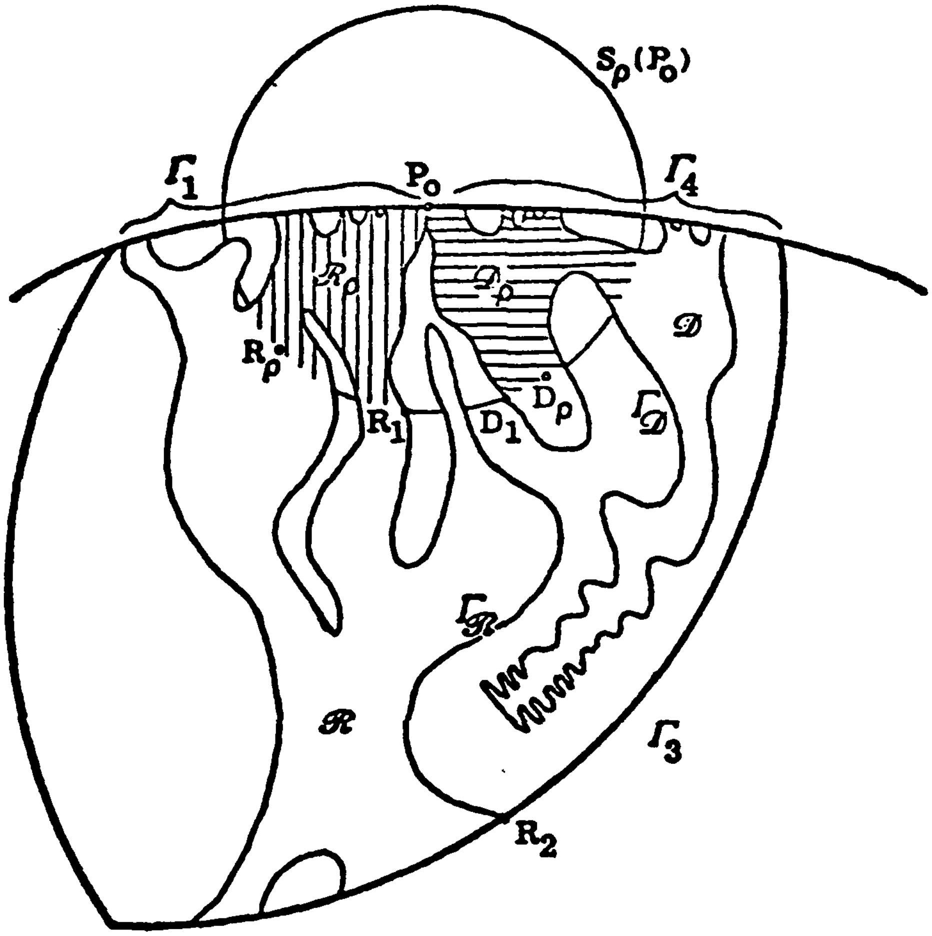

Figure 4:

$\mathscr{R}$ and $\mathscr{D}$ must cross but can't.

Let $\mathcal{R}_\rho$ be the component of $\mathcal{R} \cap B_\rho(P_0)$ which contains R_ρ in its closure. Let $\mathcal{D}_\rho$ be the component of $\mathcal{D} \cap B_\rho(P_0)$ containing D_ρ in its closure (figure 4).

By construction $\mathcal{R}_\rho^\epsilon \subset \mathcal{R}_\rho$ and $\mathcal{D}_\rho^\epsilon \subset \mathcal{D}_\rho$ so that

$$P_0 \in \bar{\mathcal{R}}_\rho, \quad P_0 \in \bar{\mathcal{D}}_\rho. \tag{23}$$

There are only a finite number of other components of $\mathcal{R} \cap B_\rho(P_0)$ having P_0 and points on $S_\rho(P_0) \cap \Omega$ as limit points, since $\partial\mathcal{R}$ has finite length. If any of these have limit points on $S_\rho(P_0)$ that are between R_ρ and D_ρ but nearer to D_ρ than the limit points of $\mathcal{R}_\rho$, relabel: Take the one of these with the nearest (counterclockwise from D_ρ) limit points on $S_\rho(P_0)$ and call it $\mathcal{R}_\rho$. Relabel one of its limit points on $S_\rho(P_0) \cap \mathcal{R}_\rho$ as R_ρ. (Such limit points in $\mathcal{R}_\rho$ exist because $\partial\mathcal{R}$ is transversal to S_ρ). Equation (23) still holds and we proceed.

Let R_1 be the point between R_ρ and D_ρ on $S_\rho(P_0)$ nearest D_ρ and in $\bar{\mathcal{R}}_\rho$. Let D_1 be the point between R_ρ and D_ρ on $S_\rho(P_0)$ nearest R_ρ and in $\bar{\mathcal{D}}_\rho$ (Figure 3).

We follow the curves $\Gamma_{\mathcal{R}}$ of $\partial\mathcal{R}$ through R_1 and $\Gamma_{\mathcal{D}}$ of $\partial\mathcal{D}$ through D_1. Let $\Gamma_{\mathcal{R}}^+, \Gamma_{\mathcal{D}}^+$ be the rays initially entering $B_\rho(P_0)$. Let $\Gamma_{\mathcal{R}}^-, \Gamma_{\mathcal{D}}^-$ be the rays initially leaving it.

Follow $\Gamma_{\mathcal{R}}^+$ into $B_\rho(P_0)$. It cannot intersect $S_\rho(P_0)$ again. (We use (23) and Lemma 5 repeatedly here): if it intersects S_ρ

(a) between Γ_1 and R_ρ, then $P_0 \notin \bar{R}_\rho$,

(b) between R_ρ and R_1 or at R_1 then $R_\rho \notin \bar{R}_\rho$,

(c) between R_1 and D_ρ, then R_1 is not the nearest point to D_ρ between R_ρ and D_ρ on $\bar{R}_\rho \cap S_\rho(P_0)$,

(d) between D_ρ and Γ_4, then $P_0 \notin \bar{\mathcal{D}}_\rho$.

Because ∂R has finite length, Γ_R^+ has exactly one limit point. Because ∂R is regular this point is on $\partial\Omega \cap \overline{B_\rho(P_0)}$. It must be P_0: otherwise the combination of (23) and Lemma 5 (with the other three points being R_ρ, D_ρ, and P_0) would be contradicted.

The same reasoning shows that $\Gamma_{\mathcal{D}}^+$ never intersects $S_\rho(P_0)$ after D_1, and has P_0 as its limit point.

Follow Γ_R^- out of $B_\rho(P_0)$. It too must eventually have a limit point [illegible] on $\partial(U \cap \Omega)$. Before it reaches R_2 it can intersect $S_\rho(P_0)$ a finite number of times between R_ρ and D_ρ. Any time it enters $B_\rho(P_0)$ through this arc it may:

(a) [illegible] on another arc of $S_\rho(P_0)$ between Γ_1 and R_ρ or between [illegible] and Γ_4. In either case, this would contradict the combination of (23) and Lemma 5.

(b) Not leave $B_\rho(P_0)$. Arguing as with Γ_R^+ it would follow that Γ_R^-'s limit point was P_0. If its last intersection R_3 with S_ρ was

(i) between R_ρ and R_1, then R_1 could not be in $\bar{R}_\rho$;

(ii) between R_1 and D_ρ, then the arc of Γ_R^+ from R_3 to P_0 bounds a component of R

having P_0 and points on S_ρ between R_1 and D_ρ as limit points. This contradicts the paragraph following (23).

(c) Leave $B_\rho(P_0)$ through the same arc.

Thus, case (c) happens, so $\Gamma_{\mathcal{R}}$ intersects the arc of $S_\rho(P_0)$ between R_ρ and D_ρ an odd number of times.

Consider the simple closed curve Σ: from P_0 to R_1 (along $\Gamma_{\mathcal{R}}^+$), from R_1 to R_2 (along $\Gamma_{\mathcal{R}}^-$), then clockwise along $\partial(\mathcal{U} \cap \Omega)$ back to P_0. By construction, R_ρ and hence all of $\mathcal{R}$ is interior to Σ. Thus, (22) implies that

$$R_2 \in (\Gamma_3 \cup \Gamma_4) \setminus (\Gamma_2 \cap \Gamma_3).$$

Because Σ is transverse to $S_\rho(P_0)$ and intersects the arc between R_ρ and D_ρ an odd number of times, D_ρ and hence all of $\mathcal{D}$ must be exterior to Σ. Thus, $\bar{\mathcal{D}}$ cannot intersect $\Gamma_1 \cup \Gamma_2$ and we have our contradiction to (22). Q.E.D.

§4. Related Questions

In what ways can Theorem 1 be generalized? It is clear from the proof that γ could be allowed to vary continuously along $\partial\Omega$, with $0 < \gamma(P) < \pi$. The differential equation could be generalized to

$$\operatorname{div} Tu = f(x,u), \quad f \text{ continuous in } (x,u), \ \frac{\partial f}{\partial u} > \varepsilon_0 > 0,$$

provided it was the Euler equation for a variational problem that implies the weak boundary behavior needed for the comparison principle.

Are there situations for which $\partial\Omega$ is not regular but u still is more than just continuous?

Conjecture. If Ω is (locally) convex in a neighborhood of $P_0 \in \partial\Omega$, with Lipschitz constant $L_1 < \tan \gamma$, then u is Lipschitz continuous there.

One cannot expect much more than Lipschitz continuity in this case: One can construct a convex domain in R^2 for which P_0 is the limit (from both sides of $\partial\Omega$) of points on $\partial\Omega$ at which $\partial\Omega$ is locally a convex corner. If u is C^1, then the contact angle condition on each segment of the corner uniquely determines the tangent plane at the corner. If $\partial\Omega$ does not have a tangent at P_0, the limiting values of the gradients of u at these corners will be different from each side (provided $\gamma \neq \pi/2$).

Do the results of Theorem 1 generalize to R^n? The method does not seem to.

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The author wishes to thank Robert Finn for posing the problem studied in this paper.

Appendix A: Proof of Lemma 4

We consider the function

$$(1) \qquad x = \frac{\kappa z}{2} y(y+h) + \delta(y+h)\varphi(z) = g(y,z)$$

and its graph S. We show that for a correct choice of the parameters h, δ, and the function φ, (a rotation of) S will satisfy the conditions of Lemma 4.

Require initially:

(a) $0 < \delta < h$;

(b) $|\varphi|_{C^2[-M,M]} < 1$, $\varphi'(z) < 0$

(c) $|z| < M$, $-h < y < \min(\alpha|x|,h)$,

where $\alpha > 0$ is chosen large enough to guarantee that the rotation of S needed to satisfy the contact angle conditions (ii) of Lemma 4 will keep S lying above a satisfactory neighborhood $U \cap \Omega$. For example, choose

(d) $\alpha > 2 \tan |\frac{\pi}{2} - \gamma|$.

We first find a φ and bound h so that S is a C^2 graph $z = v(x,y)$ and so that condition (iii) of Lemma 4 is satisfied. Let

$$(2) \qquad \varphi(z) = \begin{cases} \mu(z_1-z)^3 & z > z_1 = \frac{L+U}{2} + \frac{\delta_2}{2} \\ 0 & z_1 > z > z_0 = \frac{L+U}{2} - \frac{\delta_2}{2} \\ \mu(z_0-z)^3 & z_0 > z, \end{cases}$$

where $\mu > 0$ is small, so that u satisfies (b). If $(x,y,z) \in S$ and (x,y) is near P_0, (1) implies that $\varphi(z)$ is small. It follows from (2) that if (x,y) is near enough to P_0, z lies between the bounds of (iii).

The surface S is C^2. To show that it is a graph $z = v(x,y)$ it suffices to show that $\frac{\partial g}{\partial z} < 0$ in the entire region (c). Since

$$(3) \qquad g_z = (y+H)\ [\frac{\kappa}{2}y + \delta\varphi'(z)]$$

and since $\varphi'(z) < 0$, g_z is negative for $y < 0$. For $0 < y < \min(\alpha|x|,h)$ we are more careful. From (3) we want $\varphi'(z) < -\frac{\kappa y}{2\delta}$. So for $(x,y,z) \in S$, $0 < y < \min(\alpha|x|,h)$ it suffices to find

$$(4) \qquad \varphi'(z) < \frac{-\kappa\alpha|x|}{2\delta}.$$

We want (4) to be satisfied for small enough h. We treat the case $x < 0$, $z > z_1$. The case $x > 0$, $z < z_0$ is analogous. If $x = 0$, then $y < 0$). From (1) and (c) we have

$$0 > x = (y+h)\ [\frac{\kappa z}{2}\ y + \delta\varphi(z)] > 2h\ [\frac{\kappa M\alpha x}{2} + \delta\varphi(z)].$$

Thus

$$x[1 - \kappa M\alpha h] > 2h\delta\varphi(z).$$

If:

(e) $\quad h < h_0 < \frac{1}{M\kappa\alpha}$,

then

$$(5) \qquad x > 2Ch\delta\varphi(z), \qquad C = \frac{1}{1 - h_0 M\kappa\alpha}$$

Plugging (5) and the explicit form (2) of φ, into (4) gives

$$-3\mu(z_1-z)^2 < \kappa\alpha Ch(z_1-z)^3.$$

This inequality holds near $z_1 = z$. To make it true for all $z_1 < z < M$ the bound (e) on h may be lowered if necessary.

Thus, g_z is negative in the entire region (c) and we may write $S = S_v$. To calculate the mean curvature of S we return to the parameterization $x = g(y,z)$:

$$(6) \qquad \begin{aligned} r &= g_y = \kappa z(y + \tfrac{h}{2}) + \delta\varphi(z) = O(h) \\ s &= g_z = \tfrac{\kappa}{2} y(y + h) + \delta\ (y + h)\varphi'(z) = O(h^2) \\ g_{zz} &= \delta(y + h)\varphi''(z) = O(h^2) \\ g_{yy} &= \kappa z \\ g_{yz} &= \kappa(y + \tfrac{h}{2}) + \delta\varphi'(z) = O(h). \end{aligned}$$

So

$$\begin{aligned} \text{div}\ Tg(y,z) &= \frac{(1+s^2)g_{yy} + (1+r^2)g_{zz} - 2rsg_{yz}}{(1+r^2+s^2)^{3/2}} \\ &= \kappa z + O(h^2). \end{aligned}$$

Thus $\exists$ h_1 so that

(f) $\quad 0 < h < h_1$

implies condition (i):

$$|\text{div}\ Tv - \kappa v| < \kappa\delta_1.$$

It remains to verify (ii). Define Γ_2 and Γ_3 to be the sets in R^2 above which S has height M and $-M$, respectively. For small enough h it is easy to calculate that the restriction of (c) that $y < h$ is redundant; the (arrowhead-shaped) region A in R^2 above which S is a graph is bounded by the arcs Γ_2, Γ_3, and the wedge $y = \alpha|x|$.

To simplify the calculations we fix S and rotate $\partial\Omega$ about $P_0 = (0,0)$ (rather than rotating S and fixing $\partial\Omega$). We continue to use ϕ for the function describing $\partial\Omega$. At a point (x,y,z) of contact between S and $\partial\Omega \times R$, the (downward) normal to S is

$$\frac{1}{\sqrt{1+r^2+s^2}}(-1,r,s).$$

The (exterior) normal to $\partial\Omega \times R$ is

$$\frac{1}{\sqrt{1+\phi'^2}}(-\phi',1,0).$$

So their dot product is

$$Tv\cdot n = \frac{\phi'+r}{\sqrt{1+\phi'^2}\,\sqrt{1+r^2+s^2}}. \tag{7}$$

Since h is small and $r,s = O(h)$ the correct rotation of $\partial\Omega$ will make $\phi'/\sqrt{1+\phi'^2}$ vary near $\cos\gamma$: nearly a rotation of $\pi/2 - \gamma$ radians. The choice (d) on α insures that $\phi(x) < \alpha|x|$ for such rotations so that S is still defined above a suitable neighborhood in Ω; define Γ_1 and Γ_3 to be

the intersection of $\partial\Omega$ with the points in A to the left and right of $x = 0$, respectively.

From (5) of Section 3,

$$\lim_{x \to 0^-} \phi'(x) > \lim_{x \to 0^+} \phi'(x), \tag{8}$$

and by construction (see (2), (6))

$$\begin{aligned} \lim_{x \to 0^-} r &= \frac{\kappa h}{2}(z_1 - z_0) + \lim_{x \to 0^+} r \\ \lim_{x \to 0^-} s &= \lim_{x \to 0^+} s. \end{aligned} \tag{9}$$

It follows from (7)-(9) that for h small enough there is a rotation of $\partial\Omega$ making

$$\lim_{x \to 0^-} Tv \cdot n > \cos\gamma > \lim_{x \to 0^+} Tv \cdot n.$$

Make the parameter δ of (1) (which was free until now) small enough so that all x-values of $S \cap (\partial\Omega \times R)$ are near enough zero to force the conditions

$$Tv \cdot n > \cos\gamma \quad \text{on } \Gamma_1, \qquad Tv \cdot n < \cos\gamma \quad \text{on } \Gamma_3.$$

This finishes the verification of (ii) and thus of the entire Lemma 4.

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There are simple domains Ω in R^2 having re-entrant corners for which the variational solution u to the capillary problem exists, is bounded, but does not extend continuously to $\partial\Omega$. How possible is it to characterize

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20. Abstract (cont.)

domains for which u must extend continuously? This paper contains the following answer:

<u>Theorem</u> let $P_0 \in \partial\Omega$. Let u be the (variational) solution to the capillary problem in Ω with contact angle γ, $0 < \gamma < \pi/2$. If any of (i), (ii), or (iii) hold, then u extends continuously to P_0:

(i) Ω is convex in a neighborhood of P_0, $\partial\Omega$ has local Lipschitz constant $L_1 < \tan \gamma$.

(ii) $\partial\Omega$ is (locally) two C^1 curves meeting at P_0 with interior angle θ satisfying $\pi - 2\gamma < \theta < \pi$.

(iii) $\partial\Omega$ is C^1 at P_0.

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